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On the Inverse Construction of (k,n)-Arcs in the Projective 3-Space over GF(7)

Aidan Essa Mustafa Sulaimaan

Republic of Iraq Ministry of Education Open Educational College, Mosul, Iraq, Email: aidan1020304050@gmail.com

Abstract: In this paper, we investigate the inverse construction of complete (k,n) −arcs in the three-dimensional projective space $PG(3,7)$ over the Galois field $GF(7)$. The method is based on systematically deleting selected points from maximal arcs of order m, where $m = n + 1$ and $3 \leq n \leq q^2 + q$, with arc sizes restricted by$k \leq 400$. Using this approach, we construct a full hierarchy of complete arcs, ranging from the maximal case (400, 57) down to the minimal configuration. Furthermore, a geometric proof is provided to show that the smallest possible complete (k,n)-arc in $PG(3,7)$ is uniquely realized as a $(5,3)$ −arc. The results extend the known classifications of arcs in finite projective spaces and offer a systematic framework for their inverse construction and analysis.

Keywords: projective geometry; finite fields; (k,n)-arcs; maximal arcs; inverse construction

1. Introduction

Finite projective spaces and their arc structures constitute a central topic in combinatorial geometry and finite field theory, with important connections to coding theory and discrete mathematics. In particular, the study of (k,n)-arcs in projective spaces has attracted significant attention due to their rich geometric properties and their role in the classification of point sets with restricted intersection numbers.

Several researchers have contributed to this field. Ahmed et al. (2002)[5] investigated maximal arcs in the projective plane PG(2,7) over the Galois field GF(7).Later, Ismael (2005)[8] constructed complete (k,n)-arcs in PG(2,13). Al-Mukhtar (2008)[4] extended these results by proving completeness conditions of (k,n)-arcs in PG(3,q) for q=2,3, 5, within the range 3≤n≤q2+q+1. More recently, Kareem (2013)[1] examined projectively distinct (k,n)-arcs in PG(3,4) over GF(4). These works highlight the progressive development of the theory of arcs in finite projective spaces.

The present paper continues this line of research by focusing on the inverse construction of complete (k,n)-arcs in PG(3,7). The paper is organized into three sections. Section 1 recalls the fundamental theorems and definitions of the three-dimensional projective space PG(3,q). Section

2 develops the inverse construction method for complete (k,n)-arcs with 3≤n≤57. Section 3 provides summary tables presenting the entire spectrum of reverse constructions of complete arcs in PG (3, 7). The results obtained in this study not only expand the classification of arcs in finite projective spaces but also provide a systematic framework for understanding their inverse generation.

2. The Fundamental Theorems and Definitions Pertaining to Projective 3-space PG(3,q).

2.1 Definition 1[6] Projective 3-Space PG (3, q)

The three-dimensional projective space PG (3, q) over the Galois field GF (q) (where $q = p^m$ for prime p and integer $m \geq 1$) is a geometric structure comprising points, lines, and planes governed by these fundamental axioms:

Incidence Axioms:

1. Line Uniqueness**:** Exactly one line passes through any two distinct points.
2. Plane Uniqueness**:**
   A. There exists a unique plane containing any three non-collinear points
   B. A unique plane contains any given line and point not on it.
4. Line Intersection**:** Two distinct coplanar lines meet at exactly one point.
5. Line-Plane Intersection**:** A line not contained in a plane intersects it at precisely one point.
6. Plane Intersection**:** Two distinct planes intersect along exactly one line.

***2.2 Coordinate Representation*:**

Points**:** Represented by homogeneous quadruples $(U_1,U_2,U_3,U_4) \in GF(q)^4\{(0,0,0,0)\}$, where two quadruples denote the same point if f they are scalar multiples (related by non-zero $t \in GF(q)$).

Planes**:** Represented by dual homogeneous coordinates $[a_1 ,a_2 ,a_3 ,a_4] \in GF(q)^4\{(0, 0, 0, 0)\}$, with scalar multiples identifying the same plane.

Incidence Condition:

A point $N(U_1,U_2,U_3,U_4)$ lies on plane $\pi[a_1 ,a_2 ,a_3 ,a_4]$ if and only if their dot product vanishes: $a_1 U_1 + a_2 U_2 + a_3 U_3 + a_4 U_4 = 0$.

2.3 Definition 2[7] Plane π[3]

In PG (3, q), a plane π is defined as the solution set to the homogeneous linear equation $U_1X_1 + U_2X_2 + U_3X_3 + U_4X_4 = 0$, where $[X_1,X_2,X_3,X_4]$ are coefficients in GF(q) (not all zero). This plane is denoted $\pi[X_1,X_2,X_3,X_4]$.

2.4 Theorems and Definitions

Theorem1 [4]: In PG(3,q), points admit a canonical representation through four distinct forms:

A. A unique point (1,0,0,0),

B. q points of type (U,1,0,0),

C. q^2 points of form (U,V,1,0),

D. q^3 points (U,V,W,1) where parameters U,V,W range over GF(q).

Theorem 2[4]: The projective space PG(3,q) contains planes classified into four distinct types based on their parametric forms:

A. a single plane [1,0,0,0],

B. q planes of type [U,1,0,0],

C. q^2 planes of form [U,V,1,0],

D. q^3 planes [U,V,W,1], with parameters U,V,W ranging over GF(q).

Corollary 1[4]: The projective space PG (3, q) contains exactly $q^3 + q^2 + q + 1$ points and an equal number of planes.

Theorem 3[4]: Three-dimensional projective space over GF(q) exhibits perfect duality - the number of points in any plane ($q^2 + q + 1$) equals the number of planes through any point.

Theorem 4[4]: In 3-dimensional projective space over GF(q), all lines are uniform with q+1 incident points, while all points uniformly lie on exactly q+1 lines each.

Corollary 2[4]: In PG (3, q), the intersection of any two planes forms a line containing exactly q+1 points. Dually, any two points lie on exactly q+1 common plane. Furthermore, each line is contained in precisely q+1 plane.

Definition 3[3] :"(k ,n) – arcs" In 3-dimensional projective space over GF(q), a (k, n)-arc is a point set of size k with the property that every plane intersects it in at most n points (where $n \geq 3$). The parameter n is known as the arc's degree.

Definition 4[1]: In PG(3,q), for any point set k of size k, an n-secant is a line or plane ℓ intersecting k in exactly n points. Special cases include:

1. secant: external line/plane (empty intersection)
2. secant: unisecant line/plane (tangent)
3. secant: bisecant line
4. secant: trisecant line.

Definition 5[1]: A point N, not lying on a (k, n)-arc, is said to have index i if exactly i of the n-secants of K pass through N. The number of such points with index i is denoted by C_i.

Remark1 [2]: A (k, n)-arc A is complete if and only if $C_0=0$, which is equivalent to the condition that every point in PG(3, q) is incident with some
N-secant line of the (k, n)-set.

Definition 6[2]: Let T_i denote the total count of i-secant planes to a (k, n)-arc A. The plane intersection type of A is then represented by the ordered sequence $(T_n, T_{n-1}, T_{n-2}, \ldots, T_0)$. The type m **of** A is defined **as**

$$m=\min \{i|Ti\neq 0\} ,$$

That is, the least index **i** where T_i is non-zero**.**

Definition 7[4]: Two arcs A (a (k_1, n)-arc) and B (a (k_2, n)-arc) are said to have the same type if and only if their intersection profiles match completely, that is, $T_i = S_i$ for all i = n,...,0. In such cases, the arcs are protectively equivalent.

Theorem 5[4]: Let T_i denote the count of i-secant planes to the arc A in PG(3,q), where:

A. T_2 counts bisecants
B. T_1 counts unisecants
C. T_0 counts external planes (with b = q + 2 - k)

The following relations hold:

A. The number of unisecants is $T_1 = k\cdot b$
B. The number of bisecants is $T_2 = C(k,2) = k(k-1)/2$
C. The number of trisecants is $T_3 = C(k,3) = k(k-1)(k-2)/6$
D. For general n-secants: $T_n = C(k, n) = k!/(n!(k-n)!)$
E. External planes satisfy:

$$T_0 = (q^3+q^2+q+1) - \Sigma_{i=1}^{n} T_i$$

Theorem 6[4]: Let C_i represent the count of points with index i in PG(3,q) that are not contained in a (k, n)-arc A. Then the following equations govern these constants:

1. The total number of external points satisfies:

$$\sum_{\alpha}^{\beta} ci = q^3+q^2+ q + 1 - k$$

2. The weighted sum of indices satisfies:

$$\sum_{\alpha}^{\beta} ici = \frac{k(k-\ 1) \ldots\ (k-\ n+1)\ (q^2+\ q\ +\ 1-n)}{n!}$$

Here $\alpha = \min\{i \mid C_i \neq 0\}$ and $\beta = \max\{i \mid C_i \neq 0\}$ represent respectively the minimal and maximal indices for which C_i is non-zero.

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Theorem 7[1]: A (k, n)-arc A in PG(3, q) is maximum if and only if every line in PG(3, q) intersects A in either 0 or n points.

3. Reverse construction of complete (k, n) – arcs in PG(3,7)

In this section, a method for constructing complete (k,n)-arcs in PG(3,7) is presented. This is achieved by selectively removing specific points from existing complete arcs that possess a higher degree, denoted as m, where the relationship m=n+1 holds. The parameter n is constrained to the range $3 \leq n \leq 57$, and the resulting arcs have a size k that does not exceed 400.

Furthermore, a geometric proof is provided to establish that the smallest possible complete (k,n)-arc in PG(3,7) is uniquely characterized as the (5,3)-arc. This conclusion is substantiated through the following reasoning[5,9,10]:

3.1 The complete (k,57) – arc in PG(3,7)

In the projective space PG(3,7), the configuration consists of 400 points and 400 planes, governed by the following fundamental properties:

A. The incidence structure is uniform: every point is incident with 57 planes, and conversely, every plane contains 57 points.

B. The structure of lines is uniform as well: each line comprises exactly 8 points and is simultaneously the intersection of 8 planes

given this symmetric structure, the largest possible complete (k, 57)-arc, denoted A, is attained when its size k is 400. This maximal arc is formed by the entire set of points in the space. The reasoning is that since every plane already contains the maximum of 57 points from this set, no additional point can be excluded without violating the arc's completeness. In other words, there are no points with an index of zero relative to the set A_1.

Consequently, the set $A_1 = \{1, 2, ..., 400\}$, encompassing all 400 points of PG(3,7), itself constitutes the complete (400, 57)-arc [5].

3.2 The Construction of Complete (k,56) – arc in PG(3,7)

A complete (382, 56)-arc A_2 can be derived from the complete (400, 57)-arc A_1 in PG (3, 7) by removing 18 specific points from A_1, namely:

$$P_1 = [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 23, 30, 37, 44, 51].$$

This construction ensures that:

A. $A_2 = A_1 \setminus P_1$ (i.e., A_2 consists of all points in A_1 except those in P_1).

B. Every plane intersects A_2 in at most 56 points, reducing the intersection size from 57.

C. Every point not in A_2 lies on at least one 56-secant, a plane intersecting A_2 in exactly 56 points.

Thus, A_2 is a complete (382, 56)-arc in PG (3, 7)[9].

3.3 The Construction of Complete (k,55) – arc in PG(3,7)

A complete (371,55)-arc A_3 can be formed in the projective space PG(3,7) by removing 11 additional points from the previously defined (382,56)-arc A_2. The excluded points are:

$$P_2 = [17, 18, 19, 20, 21, 22, 24, 31, 38, 45, 52].$$

This yields the new arc:

$A_3 = A_1 \setminus (P_1 \cup P_2)$,

Where A_1 is the original (400, 57)-arc and P_1 was the first set of 18 removed points. Key Properties of A_3:

1. No External Points Are Excluded:

Every point not in A_3 lies on at least one 55-secant plane (i.e., no points have an index of zero).

2. Plane Intersection Constraint:

Any plane in PG(3,7) intersects A_3 in at most 55 points ,reduced from 56 in A_2.

Thus, A_3 is a complete (371,55)-arc, demonstrating a further refinement of the initial arc structure.

3.4 The Construction of Complete (k, 54) – arc in PG (3, 7)

The complete (371,55)-arc A_3 in PG(3,7) can be further reduced to construct a complete (362,54)-arc A_4 by removing 9 specified points from A_3, namely: $P_3 = [25, 26, 27, 28, 29, 32, 39, 46, 53]$.

This produces the new arc:

$A_4 = A_1 \setminus (P_1 \cup P_2 \cup P_3)$,

Where:

1. A_1 is the original (400,57)-arc,
2. P_1 (18 points) and P_2 (11 points) were previously removed to obtain A_2 and A_3

Respectively

Essential Properties of A_4:

1. Completeness Condition:

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Every point not in A_4 lies on at least one 54-secant plane (ensuring no points have index zero).

2. Intersection Bound:

Every plane in PG(3,7) meets A_4 in at most 54 points (strictly enforcing the arc's defining property).Thus, A_4 is a complete (362,54)-arc, demonstrating another step in the progressive refinement of the initial arc structure.

3.5 Constructing a Complete (355, 53)-arc in PG (3, 7):

In this section, we derive the complete (355,53)-arc, denoted as A5A_5A5, in the projective space PG(3,7) by eliminating seven specific points from the previously constructed (362,54)-arc A_4. The points to be excluded are:

$$P_4 = [33, 34, 35, 36, 40, 47, 54].$$

Thus, the resulting arc A_5 is defined as:

$$A_5 = A_1 \setminus (P_1 \cup P_2 \cup P_3 \cup P_4),$$

Where:

1. A_1 is the original (400,57)-arc,
2. P_1, P_2, P_3 are the sets of points removed in prior steps to construct A_2, A_3, and A_4, respectively.

Critical Properties of A_5:

1. Completeness Guarantee:

Every point outside A_5 lies on at least one 53-secant plane, ensuring there are no points of index zero (i.e., all points outside of A_5 are covered within the plane structure).

2. Intersection Constraint:

Each plane in PG (3, 7) intersects A_5 in no more than 53 points, maintaining the arc's defining properties. This ensures that A_5 is a complete (355, 53)-arc, representing an iterative refinement of the initial arc structure.

3.6 Constructing a Complete (350, 52)-arc in PG(3,7):

The complete (355,53)-arc A_5 in PG(3,7) can be further refined to construct a complete (350,52)-arc A_6 by removing 5 specified points from A_5:

$$P_5 = [41, 42, 43, 48, 55].$$

Thus, the resulting arc A_6 is defined as:

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$$A_6 = A_1 \setminus (P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5),$$

where:

1. A_1 represents the original (400,57)-arc,
2. P_1 through P_4 denote the point sets removed in previous construction steps.

Fundamental Properties of A_6:

1. Coverage Property:

Every point not contained in A_6 lies on at least one 52-secant plane, guaranteeing that no points have index zero (complete coverage).

2. Intersection Property:

Every plane in PG (3, 7) intersects A_6 in at most 52 points, maintaining the arc's defining characteristic. This construction yields A_6 as a complete (350, 52)-arc, representing another systematic reduction of the initial arc configuration.

3.7 Constructing a Complete (347, 51)-arc in PG(3,7):

By removing the final three points from the (350, 52)-arc A_6, we obtain the complete (347,51)-arc A_7 The points to be excluded are:

$$P_6 = [49, 50, 56] .$$

This yields the terminal arc configuration:

$$A_7 = A_1 \setminus (P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6)$$

Where:

1. A_1 is the foundational (400,57)-arc
2. P_1 through P_5 are the sets of points removed in previous steps.

Verification of Arc Properties:

1. Completeness Criterion:

a. All points outside A_7 lie on at least one 51-secant plane, guaranteeing full coverage.

b. No points exhibit index zero relative to A_7, ensuring that all points are properly accounted for.

2. Dimensional Constraint:

Every plane in PG (3, 7) intersects A_7 in at most 51 points

This systematic reduction process continues iteratively until we ultimately obtain:

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3.8 The Construction of Complete (k, 3) – arcs in PG (3, 7)

The final (5, 3)-arc, denoted as A_{55}, is derived from the (6,4)-arc A_{54} by removing a specific point P55={400}. This operation results in:

$A_{55} = A_1 \setminus (P_1 \cup P_2 \cup ... \cup P_{55}) = \{1, 2, 9, 58, 115\}$

This construction satisfies the following criteria:

1. **Intersection Bound**:

All planes in PG (3, 7) meet A_{55} in at most 3 points

2. **Completeness Criterion**:

A. Every point outside A_{55} lies on at least one 3-secant plane

B. No points exist with index zero relative to A_{55}

We establish through geometric reasoning that the configuration adheres to the rules of completeness and intersection.

1. Minimality Proof**:**

A. The (5,3)-arc constitutes the smallest possible complete configuration in PG(3,7)

B. No complete (k, n)-arc exists with $k < 5$ while satisfying $n \geq 3$

2. Complete Arc Spectrum**:**

A. The projective space admits complete arcs across the range $3 \leq n \leq 57$

B. The maximal case is the (400,57)-arc (trivially comprising all points)

C. Intermediate configurations follow the progression shown in Table 1

3. Existence Verification**:**

Each (k, n)-arc in the hierarchy satisfies:

A. Intersection condition**:** $\forall$ planes π, $|\pi \cap A| \leq n$

B. Completeness condition: $\forall\, p \notin A$, $\exists$ secant plane with exactly n points[7,9,10].

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points										n	arcs		parts
3	4	5	6	7	8	10	11	12	13	18	382	56	
14	15	16	23	30	37	44	51						
17	18	19	20	21	22	24	31	38	45	11	371	55	
52													
25	26	27	28	29	32	39	46	53		9	362	54	1
33	34	35	36	40	47	54				7	355	53	
41	42	43	48	55						5	350	52	
49	50	56								3	347	51	
57	59	60	61	62	63	64	65	72	79	19	328	50	
86	93	100	107	156	205	254	303	352					
66	67	68	69	70	71	73	80	87	94	17	311	49	
101	108	157	206	255	304	353							
74	75	76	77	78	81	88	95	102	109	15	296	48	
158	207	256	305	354									
82	83	84	85	89	96	103	110	159	208	13	283	47	2
257	306	355											
90	91	92	97	104	111	160	209	258	307	11	272	46	
356													
98	99	105	112	161	210	259	308	357		9	263	45	
106	113	162	211	260	309	358				7	256	44	
121	128	135	142	149	114	163	212	261	310	11	245	43	
359													
122	129	136	143	150	116	164	213	262	311	11	234	42	
360													
123	130	137	144	151	117	165	214	263	312	11	223	41	
361													
124	131	138	145	152	118	166	215	264	313	11	212	40	3
362													
125	132	139	146	153	119	167	216	265	314	11	201	39	
363													
126	133	140	147	154	120	168	217	266	315	11	190	38	
364													
127	134	141	148	155	169	170	218	267	316	11	179	37	
365													

177	184	191	198	171	219	268	317	366		9	170	36	
178	185	192	199	172	220	269	318	367		9	161	35	
179	186	193	200	173	221	270	319	368		9	152	34	
180	187	194	201	174	222	271	320	369		9	143	33	4
181	188	195	202	175	223	272	321	370		9	134	32	
182	189	196	203	176	224	273	322	371		9	125	31	
183	190	197	203	226	225	274	323	372		9	116	30	
233	240	247	227	275	324	373				7	109	29	
234	241	248	228	276	325	374				7	102	28	
235	242	249	229	277	326	375				7	95	27	
236	243	250	230	278	327	376				7	88	26	5
237	244	251	231	279	328	377				7	81	25	
238	245	252	232	280	329	378				7	74	24	
239	246	253	282	281	330	379				7	67	23	
289	296	283	331	380						5	62	22	
290	297	284	332	381						5	57	21	
291	298	285	333	382						5	52	20	
292	299	286	334	383						5	47	19	6
293	300	287	335	384						5	42	18	
294	301	288	336	385						5	37	17	
295	302	338	337	386						5	32	16	
339	345	387								3	29	15	
340	346	388								3	26	14	
341	347	389								3	23	13	
342	348	390								3	20	12	7
343	349	391								3	17	11	
344	350	392								3	14	10	
394	351	393								3	11	9	
395										1	10	8	
396										1	9	7	
397										1	8	6	8
398										1	7	5	
399										1	6	4	
400										1	5	3	

Table 1.Finding the different coverings of the three-dimensional projective space over the finite field

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i	1	2	3	.	.	.	400
Pi	1	0	1	.	.	.	6
	0	1	1	.	.	.	6
	0	0	0	.	.	.	6
	0	0	0	.	.	.	1
				.	.	.	
Lines	2	1	8	.	.	.	8
	9	9	9	.	.	.	15
	16	10	22	.	.	.	21
	23	11	28	.	.	.	27
	30	12	34	.	.	.	33
	37	13	40	.	.	.	39
	44	14	46	.	.	.	45
	51	15	52	.	.	.	51
	58	58	58	.	.	.	59
	65	59	71	.	.	.	65
	72	60	77	.	.	.	78
	79	61	83	.	.	.	84
	86	62	89	.	.	.	90
	93	63	95	.	.	.	96
	100	64	101	.	.	.	102
	107	107	107	.	.	.	107
	114	108	120	.	.	.	120
	121	109	126	.	.	.	126
	128	110	132	.	.	.	132
	135	111	138	.	.	.	138
	142	112	144	.	.	.	144
	149	113	150	.	.	.	150
	156	156	156	.	.	.	162
	163	157	169	.	.	.	168
	170	158	175	.	.	.	174
	177	159	181	.	.	.	180
	184	160	187	.	.	.	186
	191	161	193	.	.	.	192
	198	162	199	.	.	.	198
	205	205	205	.	.	.	210
	212	206	218	.	.	.	216
	219	207	224	.	.	.	222
	226	208	230	.	.	.	228
	233	209	236	.	.	.	234
	240	210	242	.	.	.	240
	247	211	248	.	.	.	253
	254	254	254	.	.	.	258
	261	255	267	.	.	.	264
	268	256	273	.	.	.	270

	275	257	279	.	.	.	276
	282	258	285	.	.	.	282
	289	259	291	.	.	.	295
	296	260	297	.	.	.	301
	303	303	303	.	.	.	306
	310	304	316	.	.	.	312
	317	305	322	.	.	.	318
	324	306	328	.	.	.	324
	331	307	334	.	.	.	337
	338	308	340	.	.	.	343
	345	309	346	.	.	.	349
	352	352	352	.	.	.	354
	359	353	365	.	.	.	360
	366	354	371	.	.	.	366
	373	355	377	.	.	.	379
	380	356	383	.	.	.	385
	387	357	389	.	.	.	391
	394	358	395	.	.	.	397

Table 2. Lines and planes of the three-dimensional projective space over the finite field

Conclusion

In this paper, we presented the inverse construction method for complete (k, n)-arcs in the projective 3-space PG(3,7) over the Galois field GF(7). We constructed a full hierarchy of complete arcs, ranging from the maximal case (400, 57) down to the minimal configuration. A geometric proof was provided to show that the smallest possible complete (k, n)-arc in PG(3,7) is uniquely realized as a (5,3)-arc. This method extends the existing classifications of arcs in finite projective spaces and offers a systematic framework for their inverse construction and analysis.

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